Analog passive Filters

I. INTRODUCTION

Passive analog filters are basic components of signal processing widely used in electronics. Their many applications include the separation of an audio signal before applying it to a bass, mid and treble speaker, the combination and subsequent separation of several telephone conversations on one channel, the selection of a radio station selected from one radio receiver and the rejection of the others.

A filter is therefore a circuit whose behavior depends on the frequency of the signals. Therefore, it must contain at least one capacitor or coil. Filters are distinguished:

- low pass (cut off high frequencies),
- high pass (cut off the low frequencies),
- band-pass that only allow a defined frequency band to be passed,
- Notch band or cut band.

II. Transfer Functions (TF)

1. Definitions

One can symbolize a filter by the following scheme:



If $v_e(t) = V_E.\sqrt{2}sin(\omega t)$ then $\underline{V_E} = V_E$

- V_E is constant: it is the RMS value of the input signal and it is fixed no matter what the signal frequency is.
- The argument of $\underline{V_E}$ is null by definition : the input signal is the reference against which the phase shift of the output signal is calculated.

The transfer function (TF) $\underline{H}(\omega)$ of the filter is defined as:

$$\underline{H}(\omega) = \frac{\underline{V_S}}{\underline{V_e}}$$

- $\left|\underline{H}(\omega)\right| = \frac{V_S}{V_e}$ is the module of the TF, it corresponds to the ratio of the RMS value of the output signal over the RMS value of the input signal. $\left|\underline{H}(\omega)\right|$ is the gain and is also denoted as $G(\omega)$. It's a unit less value and it is necessarily positive or null.
- $\varphi(\omega) = Arg\,(\underline{H}(\omega)) = Arg\,(num) - Arg\,(dén) = \varphi - 0 = \varphi$: The argument of the TF corresponds to the phase shift of the output signal with respect to the input signal.

  $\varphi(\omega)$ is expressed either in radians or better in degrees (directly measurable on an oscilloscope). $\varphi(\omega)$ is positive if the output signal leads the input signal and negative otherwise.

To calculate the TF, we use the fundamental laws (Ohm's law, Kirchhoff's law, Thévenin or Norton theorems, Millman's theorem, voltage divider formula, etc...) remembering that the complex impedance of a capacitor is $1/jC\omega$ and of a coil is $jL\omega$.

2. Study of the module (or Gain $G(\omega)$)

a) Filter order

When calculating the FT, we obtain a complex polynomial or a quotient of complex polynomials in which terms appear in ω, ω^2 … : the order of the filter corresponds to the degree of the polynomial, ie to the exponent of the higher degree term in ω. We will limit the study to filters of order 2.

Physically, the filter order corresponds to the number of non-resistive components (coils or capacitors) of the circuit.

b) Filter type

Amplification varies according to the frequency, the study of this variation is conducted as the classical study of a function $y = f(x)$, x being the pulsation ω and y the gain: study of the derivative, search for extrema, variation direction, particular values.

Upon the variation direction, we set 4 **types** of filters:

- $G(\omega)$ increases when f increases: high pass filter
- $G(\omega)$ decreases when f increases: low pass filter
- $G(\omega)$ increases then decreases when f increases: band pass filter (necessarily of order>1)
- $G(\omega)$ decreases then increases when f increases: notch (band-reject) filter (necessarily of order >1)

Note : for first order filters,$G(\omega)$ is necessarily monotonous (increasing or decreasing).

c) Gain in decibel

The filter gain in decibel is equal to 20 times the **decimal logarithm** of the gain: it is expressed in decibels (symbol : dB).

Therefore, $G_{db}(\omega) = 20.log(G(\omega))$ or $20.log(|\underline{H}(\omega)|)$

Note 1: "log" function being strictly monotonous and increasing, the variation direction of the Gain is obviously the same as that of the amplification.

Note 2: The decibel is in fact a pseudo-unit, since $H(\omega)$ has no unit (one cannot take the logarithm of a quantity that has a physical unit).

Note 3 : Why work with $G_{dB}(\omega)$rather than with $G(\omega)$?

- *The logarithmic function reduces the range of variation of a quantity; therefore, both small and large values of $G(\omega)$can be studied on the same graph.*
- *The eyes (natural sensors of light intensity) and the ears (natural sensors of acoustic intensity) are more sensitive to a logarithmic scale than to a linear scale.*

We will look for the value of the maximum gain in decibel ($G_{dB_{Max}}$) which is an important parameter of the filter, as well as the frequency for which we obtain this maximum gain in decibel.

If the gain (in decibels) and the phase shift are plotted on semi-logarithmic paper, the resulting graphs are called **Bode diagrams**. Examples of such plots are provided in the appendix.

For further study

d) Tuning Frequency (or central frequency)

*For a Band pass filter (or notch filter) the frequency for which the Gain is maximum (or minimum) is called the **tuning or central frequency**: it's one of the most important parameters characterizing a filter.*

e) Cut-off frequency

The cut-off frequency f_C is the frequency at which filter gain in decibel has fallen by 3 dB with respect to the maximum gain in decibel, in other words, it's the frequency at which the gain is divided by $\sqrt{2}$ with respect to the maximum gain (the second definition is more convenient for calculations).

***Therefore : for $f = f_C$, $G_C = G_{dB_{MAX}} - 3\,dB$ or $G_C = G_{MAX}/\sqrt{2}$** (the index c stands for Cut)*

Note 1: for low-pass and high-pass filters, there is only a single cut-off frequency. For band-pass and notch filters there are two cut-off frequencies called low cut-off frequency (f_{cB}) and high cut-off frequency ($f_{cH} > f_{cB}$).

Note 2: contrary to the idea contained in the word cut, it should not be said (or believed ...) that beyond the cut-off frequency, the signal no longer passes: it is only increasingly attenuated, but it does not disappear abruptly.

f) Bandwidth

This is the frequency range between the two cutoff frequencies:

Bandwidth – for bandpass and bandcut filters – is equal to the difference $f_{cH} - f_{cB}$.

Note: the selectivity of a band-pass or band-reject filter is measured by the ratio: center frequency over bandwidth: the greater the ratio, the more selective the filter.

g) Filter Slope and Order

We are also interested in the "speed" at which decreases the gain in decibel $G_{dB}(\omega)$ *outside the bandwidth. The faster it decreases, the more effectively the filter attenuates signals outside the bandwidth (which is a priori the useful area)*

In the expression of gain in decibel, we calculate the linear asymptotes when ω *tends towards 0 then towards infinity: we trace these asymptotes. The slope can then be calculated.*

This slope is expressed in decibels per decade (10-fold change in frequency)

The slope is related to the filter ***order****. For instance, for low-pass and high-pass filters, we have:*

- *1st order filter: slope of the asymptote:* $\pm 20\ dB$ *per decade.*
- *2nd order filter: slope of the asymptote :* $\pm\ 40\ dB$ *per decade.*
- *Filter of order n: slope of the asymptote :* $\pm\ 20.n\ dB$ *per decade.*

The slope therefore reflects the efficiency with which the filter attenuates unwanted signals.

h) Plotting the Gain curve

The Gain curve is plotted as a function of the frequency on semilogarithmic millimeter paper: indeed, the slope of the gain in decibel being linear with respect to the log of the frequency (when f is very far away from the bandwidth), the curve will be straight in this part of the graph.

i) Study of the argument (or Phase shift $\varphi(\omega)$ *)*

Phase shift varies according to frequency, studying this variation is conducted like that of the Gain: derivative sign, search for the extrema, variation direction, particular values.

<u>Attention</u>: it is necessary to carefully study in which dial of the complex plane the complex number representing the TF is positioned. Indeed $Arg\ (\underline{H}(\omega)) = Arctan(b/a)$ *when* $\underline{H}(\omega) = a + jb$*, but the calculator always gives the angle between* $-\frac{\pi}{2}$ *et* $+\frac{\pi}{2}$*, i.e. the calculator always "sees"* $a > 0$.

The full variation of the phase shift is related to the order n *of the filter and worth* $n.\frac{\pi}{2}$

III. Standard forms of the transfer function for filters of order 1 and 2

The transfer functions for filters of order 1 and 2 whose amplification tends towards 0 outside the bandwidth can be put in a particular form, called standard (or general) form. The form depends on:

- Filter order
- The nature of the filter

The table below gives you all the standard transfer functions you may need!

Filter type	Order 1	Order 2
Low Pass	$\underline{H}(\omega) = G_{Max}.\frac{1}{1+j\frac{\omega}{\omega_c}}$ with : $G_{Max} = G_{VLF}$ ω_c = Cut-off pulsation	$\underline{H}(\omega) = G_0.\frac{1}{1+2j\sigma\frac{\omega}{\omega_0}-\left(\frac{\omega}{\omega_0}\right)^2}$ with : $G_0 = G_{VLF}$
High Pass	$\underline{H}(\omega) = G_{Max}.\frac{j\frac{\omega}{\omega_c}}{1+j\frac{\omega}{\omega_c}}$ with : $G_{Max} = G_{VHF}$ ω_c = Cut-off pulsation	$\underline{H}(\omega) = G_0.\frac{-\left(\frac{\omega}{\omega_0}\right)^2}{1+2j\sigma\frac{\omega}{\omega_0}-\left(\frac{\omega}{\omega_0}\right)^2}$ with : $G_0 = G_{VHF}$
Band Pass		$\underline{H}(\omega) = G_0.\frac{2j\sigma\frac{\omega}{\omega_0}}{1+2j\sigma\frac{\omega}{\omega_0}-\left(\frac{\omega}{\omega_0}\right)^2}$ with : $G_0 = G_{Max}$

Reminder: VLF = Very Low Frequencies ($f \to 0$);

VHF = Very High Frequencies ($f \to \infty$)

To put a transfer function in standardized form, we need to determine the following parameters:

- G_{Max} and ω_c for 1st order filters
- G_0, σ and ω_0 for 2nd order filters

Note1: These parameters are always positive real constants! They must not contain ω, j and sign -.

Note 2: For order 1 filters, you can find ω_c in 2 different ways:

Either with the standard form of the transfer function

Either using the definition $G(\omega_c) = \frac{G_{Max}}{\sqrt{2}}$

Note 3: σ is called damping factor

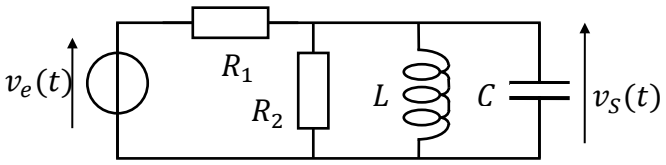
ω_0 is called natural angular frequency.

To put a filter transfer function in its generalized form, you must always start by putting the denominator in the correct form. You will notice that there is always a '1 ' in the real part of this denominator, it is what we must start by making appear. Generally, the other terms of the denominator will not be a problem.

Then work on the numerator, which has a particular shape depending on the nature of the filter. You will notice that it is always equal to one of the members of the denominator.

Finally, you can proceed by identification!

EXAMPLE:



This filter is a band pass filter ($A \to 0$ at VLF and VHF). Its transfer function is expressed as:

$$\underline{H}(\omega) = \frac{jR_2L\omega}{R_1R_2 + j(R_1 + R_2)L\omega - R_1R_2LC\omega^2}$$

First, we rewrite the denominator in order to show "1". We factorize it with R_1R_2.

$$\underline{H}(\omega) = \frac{1}{R_1R_2} \times \frac{jR_2L\omega}{1 + j\frac{R_1 + R_2}{R_1R_2}L\omega - LC\omega^2}$$

$$\Rightarrow \underline{H}(\omega) = \frac{1}{R_1} \times \frac{jL\omega}{1 + j\frac{R_1 + R_2}{R_1R_2}L\omega - LC\omega^2}$$

We notice that the intended form is actually the form of a band pass filter: the numerator is pure imaginary, however the numerator and the denominator must have the same imaginary part. We should then rewrite the transfer function by multiplying the numerator by $\frac{R_1+R_2}{R_1R_2}$:

$$\underline{H}(\omega) = \frac{1}{R_1} \times \frac{R_1R_2}{R_1 + R_2} \times \frac{\frac{R_1 + R_2}{R_1R_2}.jL\omega}{1 + j\frac{R_1 + R_2}{R_1R_2}L\omega - LC\omega^2}$$

$$\Rightarrow \underline{H}(\omega) = \frac{R_2}{R_1 + R_2} \times \frac{j\frac{R_1 + R_2}{R_1R_2}L\omega}{1 + j\frac{R_1 + R_2}{R_1R_2}L\omega - LC\omega^2}$$

Now, we can make the identification:

$$\begin{cases} G_0 = \dfrac{R_2}{R_1 + R_2} \\ \left(\dfrac{\omega}{\omega_0}\right)^2 = LC\omega^2 \;\Rightarrow \omega_0 = \dfrac{1}{\sqrt{LC}} \\ 2\sigma\dfrac{\omega}{\omega_0} = \dfrac{R_1 + R_2}{R_1 R_2} L\omega \;\Rightarrow \sigma = \dfrac{1}{2} \times \dfrac{R_1 + R_2}{R_1 R_2} L\omega_0 \end{cases}$$

IV. Summary: FILTER STUDY STEPS

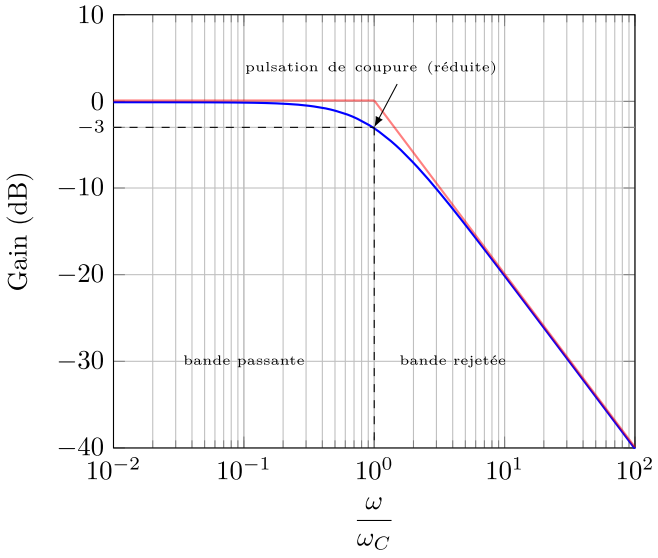
- Determine the filter order: it corresponds to the number of non-resistive components.
- Determine the filter type: imaginer the input signal frequency tending towards 0 then towards infinity and calculate the output signal amplitude using the following rules:
  - ✓ In Very Low Frequencies (VLF), $f \to 0 \Rightarrow |Z_C| \to \infty$: replace the capacitor with an open switch. (remove it !)
  - ✓ In Very Low Frequencies (VLF), $f \to 0 \Rightarrow |Z_L| \to 0$: replace the coil with a wire
  - ✓ In Very High Frequencies (VHF), $f \to \infty \Rightarrow |Z_C| \to 0$: replace the capacitor with a wire
  - ✓ In Very High Frequencies (VHF), $f \to \infty \Rightarrow |Z_L| \to \infty$: replace the coil with an open switch. (remove it !)

  Once the rules are applied, only resistors and input generator remain in the circuit: calculating the output signal amplitude is therefore very simple.
- Calculate the TF and write it in standard form (which will be mentioned later).
- Find the relations between the filter components and its parameters (maximum Gain, cut-off frequency(ies), Bandwidth, etc.…).
- Calculate the value of each component in order to obtain desired filter parameters.

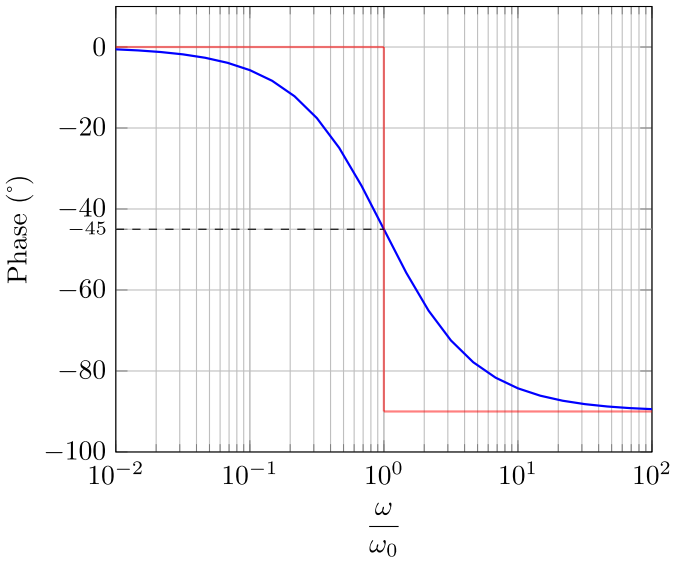
ANNEXES – Bode Diagrams

✓ Low pass filter

- Gain curve

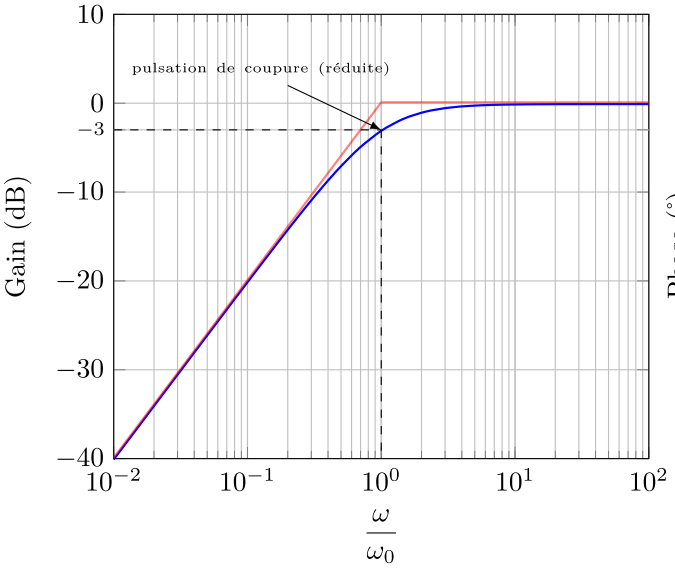


- Phase curve

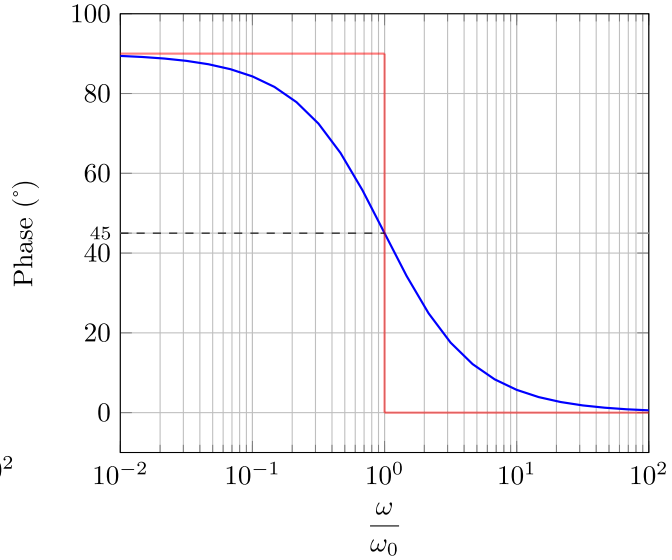


✓ High Pass filter

- Gain curve

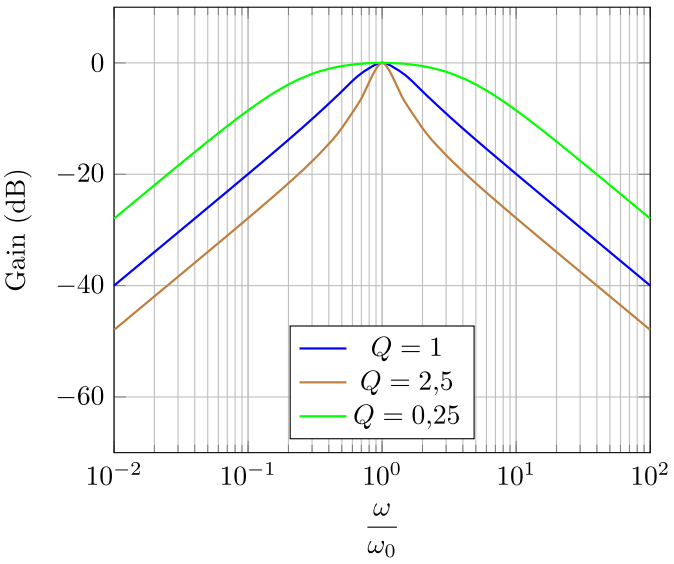


- Phase curve

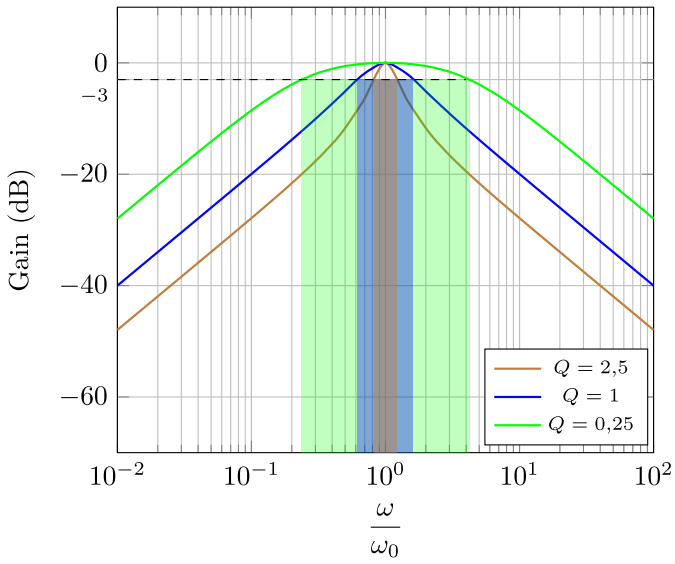


✓ Bans pass filter

- Gain curve



- Phase curve



Rq : $Q = 1/2\sigma$